FINDING A NASH EQUILIBRIUM IS NO EASIER THAN BREAKING FIAT-SHAMIR

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Today

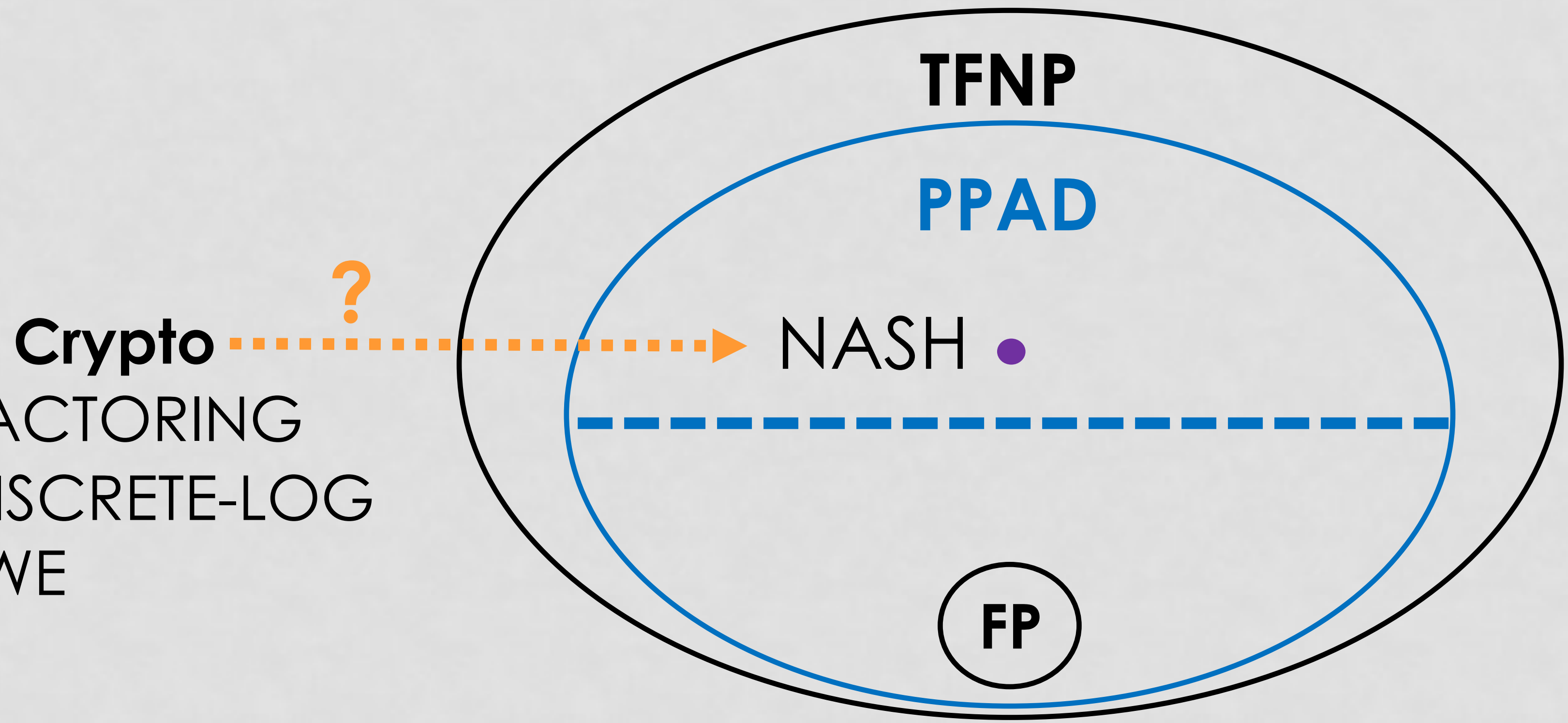
- Average-case hardness in PPAD

- Theorem: PPAD is as hard as breaking soundness of Fiat-Shamir when applied to the sumcheck protocol
- Corollary: Average-case hardness in PPAD relative to a random oracle

- Result extends to CLS $\subseteq$ PPAD

NASH and PPAD

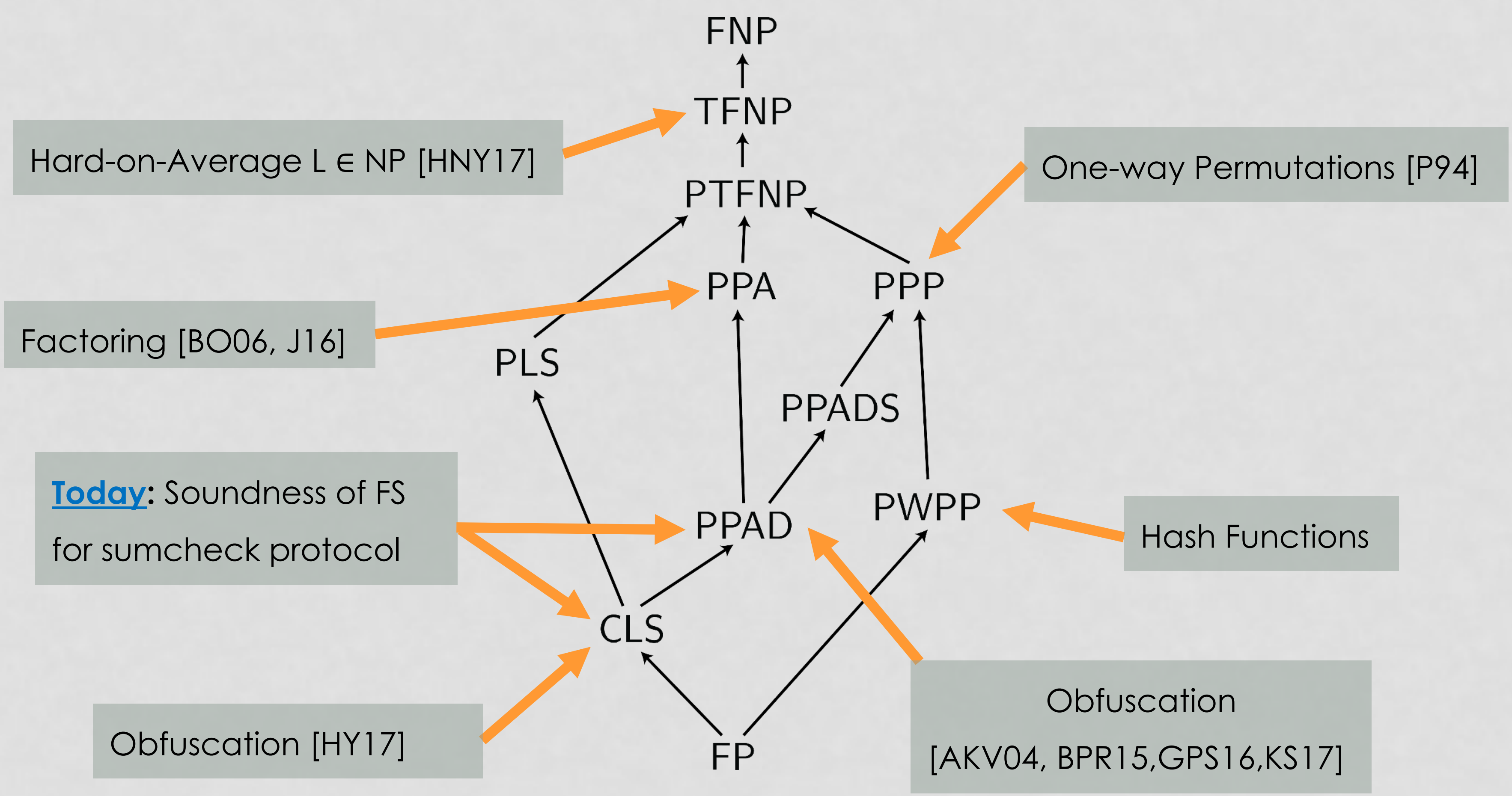
[P94,DGP05,CDT09]



- FACTORING
- DISCRETE-LOG
- LWE

- **T**otal **F**unctional **NP**
- Totality via "parity argument in directed graphs"

Average-case hardness in TFNP



PPAD-Hardness from Obfuscation

[BPR15]

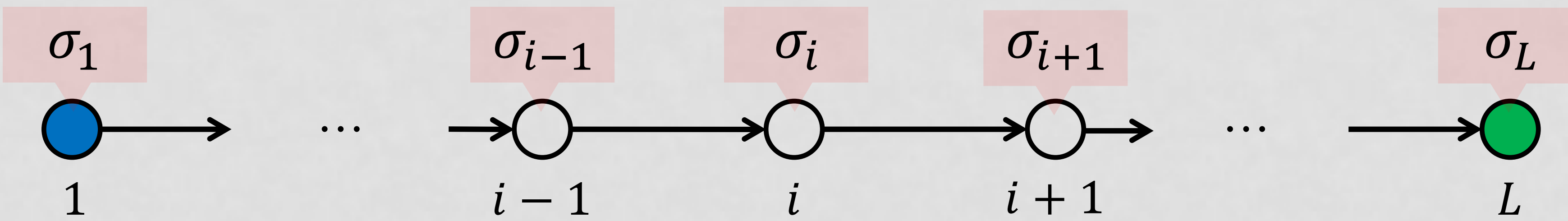
<u>Theorem</u>: Indistinguishability obfuscation (iO) implies hardness in PPAD/CLS

1. iO ⟶ SINK-OF-VERIFIABLE-LINE (SVL)
2. SVL ⟶ NASH (END−OF−LINE) [AKV04]

2.* SVL ⟶ END-OF-METERED-LINE ($\in$ CLS) [HY17]

<u>Bottom-line</u>: Focus on hard SVL instances

SINK-OF-VERIFIABLE-LINE (SVL)
[AKV04, BPR15]

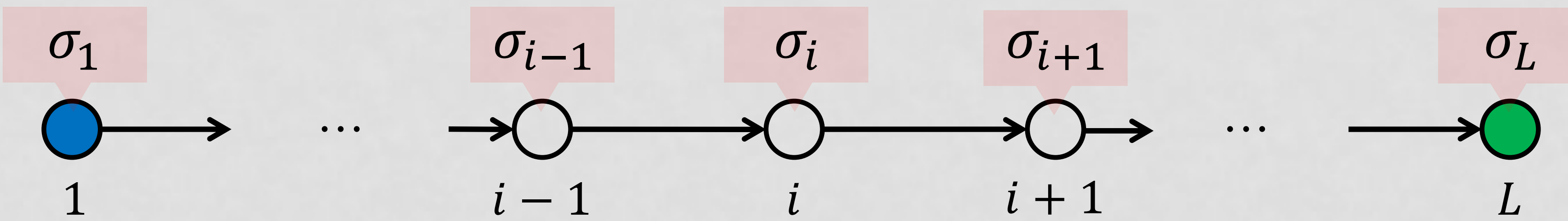


Exponential-sized graph with vertices in $\{0,1\}^n$

- Path defined by circuit $\boldsymbol{S}: \{0,1\}^n \rightarrow \{0,1\}^n$
- Verifier circuit $\boldsymbol{V}: [2^n] \times \{0,1\}^n \rightarrow \texttt{ACCEPT/REJECT}$
- Promise: $\boldsymbol{V}(i, \sigma_i) = \texttt{ACCEPT} \Leftrightarrow \sigma_i = \boldsymbol{S}^i(\sigma_1)$
- Solution: $\sigma_L = \boldsymbol{S}^L(\sigma_1)$

SVL IS NO EASIER THAN BREAKING FIAT-SHAMIR

SVL as Verifiable Counter for #SAT



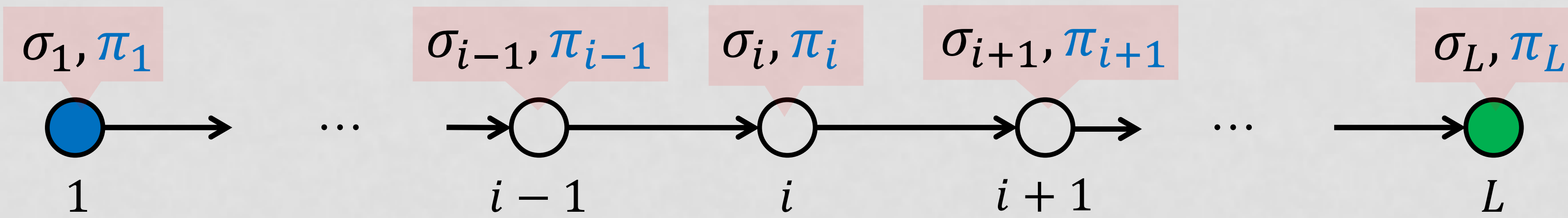
Reduce #SAT to SVL

- $\varphi(z_1, \ldots, z_n) \mapsto (\boldsymbol{S}, \boldsymbol{V}, L \leftarrow 2^n)$
- $\sigma_i \leftarrow$ # of satisfying assignments between 0^n and i

<u>Challenge</u>: How to verify σ_i?

<u>Solution</u>: Append a succinct proof π_i

SVL as Verifiable Counter for #SAT

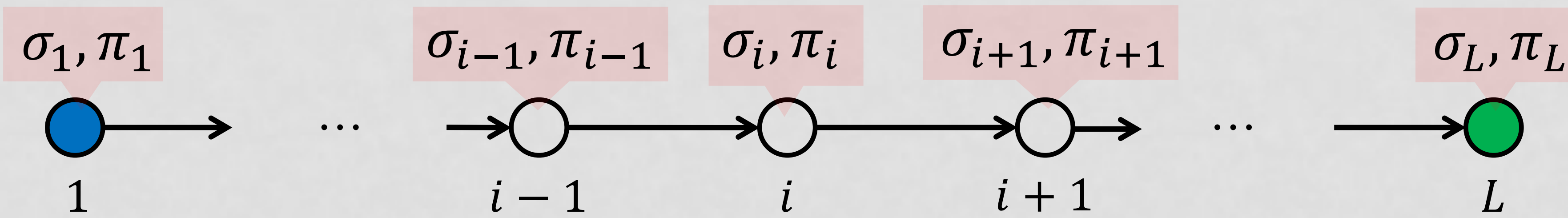


$$\boldsymbol{V}(i, \sigma_i, \pi_i) = \texttt{ACCEPT}$$

$$\Updownarrow$$

σ_i is the number of satisfying assignments between 0^n and i

SVL as Verifiable Counter for #SAT



Challenge: getting π_i to be of size $poly(n)$

Solution: use sumcheck protocol [LFKN92]

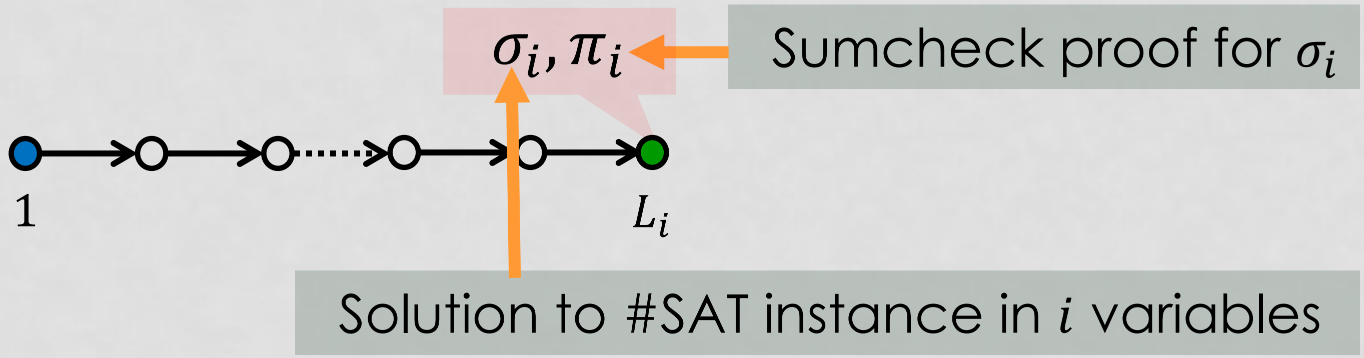
Challenge: protocol is interactive

Solution: Fiat-Shamir transform [FS86]

Challenge: computing $S(\sigma_i, \pi_i) = (\sigma_{i+1}, \pi_{i+1})$

Solution: incremental proof update

Recursive Approach



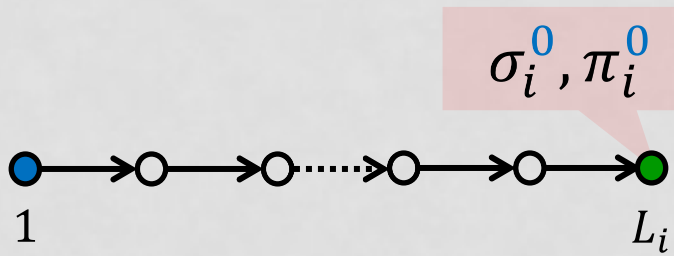
SVL counter $(\boldsymbol{S}_i, \boldsymbol{V}_i, L_i)$ for $\varphi(z_1, \ldots, z_i)$

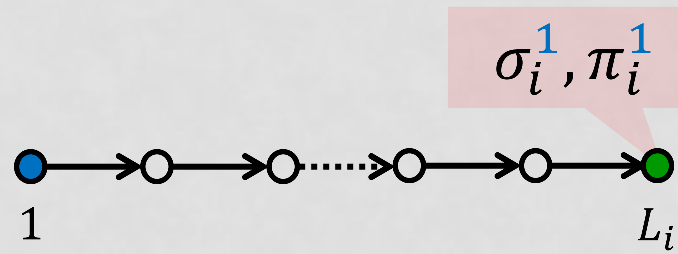
$$\Downarrow$$

SVL counter $(\boldsymbol{S}_{i+1}, \boldsymbol{V}_{i+1}, L_{i+1})$ for $\varphi(z_1, \ldots, z_i, z_{i+1})$

Base case: Length one, with empty proof

Naïve Construction

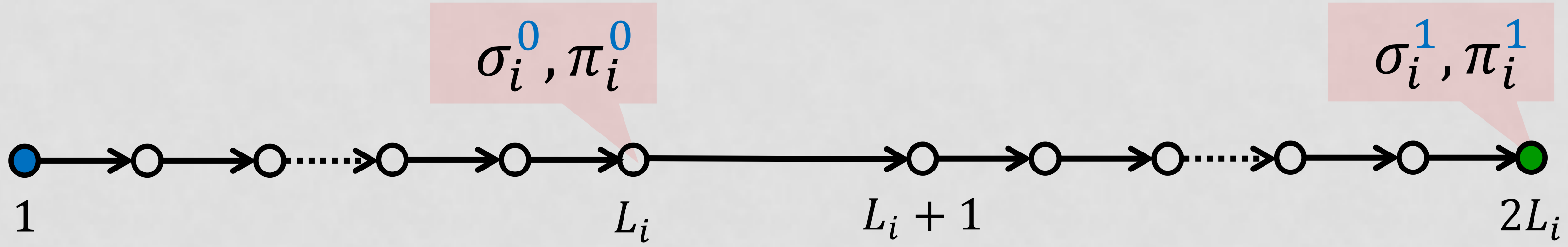




Counter for $\varphi(z_1, \ldots, z_i, z_{i+1})$

- Left path: Run counter on $\varphi(z_1, \ldots, z_i, 0)$
- Right path: Run counter on $\varphi(z_1, \ldots, z_i, 1)$

Naïve Construction



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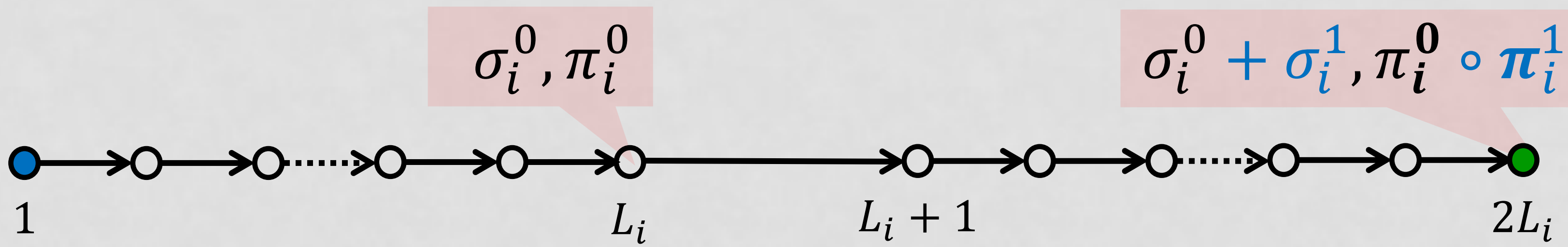


Counter for $\varphi(z_1, \dots, z_i, z_{i+1})$

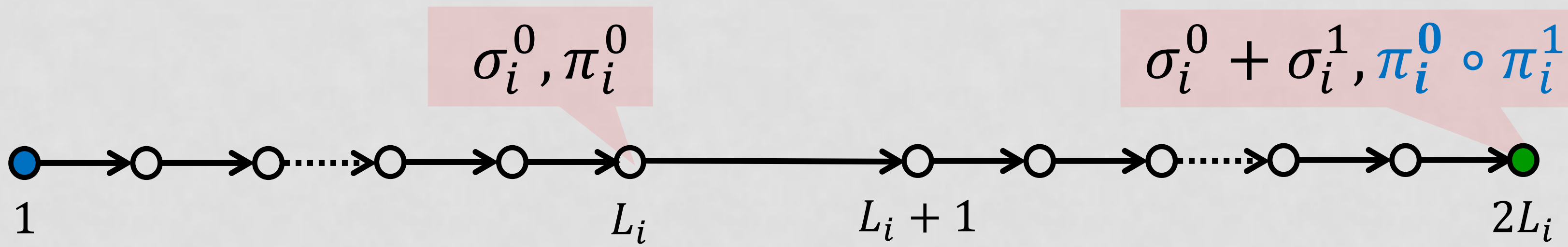
- Left path: Run counter on $\varphi(z_1, \dots, z_i, 0)$
- Right path: Run counter on $\varphi(z_1, \dots, z_i, 1)$ + update

Number of steps: $L_{i+1} = 2L_i$

Proof size: $P_{i+1} = 2P_i \implies P_n = 2^n$

Issue: exponential blow-up in proof/label size

New Idea: Incremental Merge



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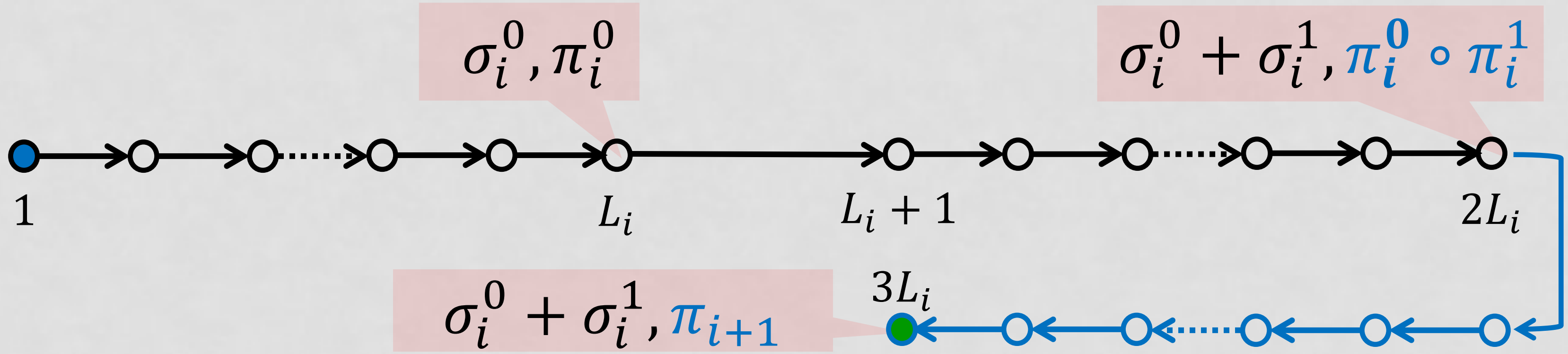
New Idea: Incremental Merge



Counter for $\varphi(z_1, \ldots, z_i, z_{i+1})$:

- Left path: Run counter on $\varphi(z_1, \ldots, z_i, 0)$
- Right path: Run counter on $\varphi(z_1, \ldots, z_i, 1)$ + updates
- Merge path: Run counter for merging $\pi_i^0 \circ \pi_i^1$ into π_{i+1}

Number of steps: $L_{i+1} = 3L_i \Longrightarrow L = L_n = 2^{n \cdot \log(3)}$

Proof size: $P_{i+1} = P_i + poly(n) \Longrightarrow P_n = poly(n)$

Fiat-Shamir for Sumcheck

Challenge: Off-path vertices due to

1. Soundness errors: accepting proof π for false statements y
2. Ambiguous proofs: accepting proof $\pi' \neq \pi$ for true statement y

Solution: Use "relaxed" SVL

Main assumption: resulting non-interactive argument is unambiguously sound for poly-time provers

Sanity check: True relative to a random oracle (and hence PPAD$\not\subseteq$FP relative to a random oracle)

Future Directions

- Instantiating Fiat-Shamir for sumcheck
  - Optimal hardness of circular-secure FHE: full version
  - From plain LWE?
- Factoring in PPAD?
  - PPAD-hardness from number-theoretic assumptions: eprint 2019/619, 2019/667
- Sampling small(ish) hard instances of NASH

THANK YOU. QUESTIONS?